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J. B. BRANDEBERRY

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APPLICATION OF GENERAL SOLUTION OF THE TWO-DIMENSIONAL PROBLEM IN POLAR COORDINATES TO STRESS ANALYSIS IN WEDGES

By

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June, 1932

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APPLICATION OF GENERAL SOLUTION OF THE TWO-DIMENSIONAL PROBLEM IN POLAR COORDINATES TO STRESS ANALYSIS IN WEDGES

The general solution of the compatibility equation in polar coordinates:[1]

$$(1)\quad \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}\right) = 0,$$

gives the stress function:

$$\begin{aligned}(2)\quad \varphi = {} & a_0 \log r + b_0 r^2 + c_0 r^2 \log r + d_0 r^2 \theta + a_0' \theta \\ & + \frac{a_1}{2} r \theta \sin\theta + (b_1 r^3 + a_1' r^{-1} + b_1' r \log r) \cos\theta \\ & - \frac{c_1}{2} r \theta \cos\theta + (d_1 r^3 + c_1' r^{-1} + d_1' r \log r) \sin\theta \\ & + \sum_{n=2}^{\infty} (a_n r^n + b_n r^{n+2} + a_n' r^{-n} + b_n' r^{-n+2}) \cos n\theta \\ & + \sum_{n=2}^{\infty} (c_n r^n + d_n r^{n+2} + c_n' r^{-n} + d_n' r^{-n+2}) \sin n\theta.\end{aligned}$$

The stresses are obtained from this stress function by differentiation, as follows:

$$(3)\quad \sigma_\theta = \frac{\partial^2 \varphi}{\partial r^2}; \qquad \sigma_r = \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2};$$

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial \varphi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \varphi}{\partial r \partial \theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \varphi}{\partial \theta}\right),$$

where σ_θ represents the tangential normal-stress component, σ_r the radial normal-stress component, and $\tau_{r\theta}$ the shearing stress component.

---

1. Equations 1, 2, 3, and 4, given below, are taken from Timoshenko, Theory of Elasticity, 1932.

We calculate the stress components, and for our general expressions, take account only of those terms which give stresses involving r to the zero degree and higher. With these limitations, the stresses assume the form:

$$
\begin{aligned}
(4)\ \sigma_\theta &= 2b_0 + 2d_0\theta + 2a_2\cos 2\theta + 2c_2\sin 2\theta \\
&\quad + 6r(b_1\cos\theta + d_1\sin\theta + a_3\cos 3\theta + c_3\sin 3\theta) \\
&\quad + 12r^2(b_2\cos 2\theta + d_2\sin 2\theta + a_4\cos 4\theta \\
&\qquad + c_4\sin 4\theta) \\
&\quad + \text{----------} + (n+2)(n+1)r^n[(b_n\cos n\theta \\
&\qquad + d_n\sin n\theta + a_{n+2}\cos(n+2)\theta \\
&\qquad\quad + c_{n+2}\sin(n+2)\theta].
\end{aligned}
$$

$$
\begin{aligned}
\tau_{r\theta} &= d_0 + 2a_2\sin 2\theta - 2c_2\cos 2\theta + 2r(b_1\sin\theta \\
&\quad - d_1\cos\theta + 3a_3\sin 3\theta - 3c_3\cos 3\theta) \\
&\quad + 6r^2(b_2\sin 2\theta - d_2\cos 2\theta + 2a_4\sin 4\theta \\
&\quad - 2c_4\cos 4\theta) \\
&\quad + \text{----------} + (n+1)r^n[n b_n\sin n\theta \\
&\quad - n d_n\cos n\theta + (n+2)a_{n+2}\sin(n+2)\theta \\
&\quad - (n+2)c_{n+2}\cos(n+2)\theta].
\end{aligned}
$$

$$
\begin{aligned}
\sigma_r &= 2b_0 + 2d_0\theta - 2a_2\cos 2\theta - 2c_2\sin 2\theta \\
&\quad + 2r(b_1\cos\theta + d_1\sin\theta - 3a_3\cos 3\theta \\
&\quad - 3c_3\sin 3\theta) - 12r^2(a_4\cos 4\theta + c_4\sin 4\theta) \\
&\quad + \text{----------}.
\end{aligned}
$$

We thus see that each power of r is associated with four arbitrary constants. Hence if the stresses applied to the boundary $\theta = \alpha$, $\theta = \beta$, are given in integral powers of r, the stresses in the wedge included between $\theta = \alpha$, $\theta = \beta$, may be found.

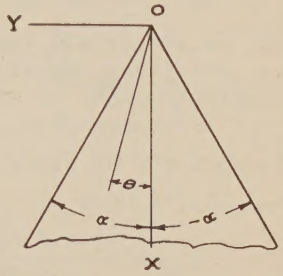

Figure 1

In the work that follows, the wedge is assumed to be of unit thickness, and the polar axis is taken as the line of symmetry of the wedge, as is shown in Figure 1. The X- and Y-axes are also shown, as in some cases it is convenient to express results in terms of stress components parallel to these axes.

We first take account of the stress function involving only terms free of r and of the first degree in r:

$$(5)\quad \varphi = a_0'\theta + \frac{a_1}{2} r\theta \sin\theta - \frac{c_1}{2} r\theta \cos\theta + b_2' \cos 2\theta + d_2' \sin 2\theta.$$

Using the second term only, we have the case of a concentrated load at the vertex of the wedge, in the direction of the axis.[2] Using only the third term, we have the case of a concentrated load at the vertex, perpendicular to the axis.[2]

The remaining three terms give the stress function:

$$(6)\quad \varphi = a_0'\theta + b_2' \cos 2\theta + d_2' \sin 2\theta,$$

and applies to the case of a moment M, acting about the vertex of the wedge.[3] This necessarily follows, since the boundary conditions for this case are

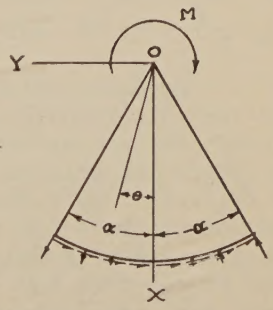

Figure 2

2. Loc. cit., page 1.
3. See Akira Miura, "Spannungskurven in rechteckigen und keilformigen Tragern," 1928.

$(\sigma_\theta)_{\theta = \pm\alpha} = (\tau_{r\theta})_{\theta = \pm\alpha} = 0$, and $\sigma_\theta = \frac{\partial^2 \varphi}{\partial r^2}$ precludes terms of the second degree or higher in r. Furthermore, the first degree terms have already been used for concentrated load at the vertex.

From symmetry, the radial stress will equal zero when $\theta = 0$; since $\sigma_r = \frac{1}{r^2}(-4\ b_2' \cos 2\theta - 4\ d_2' \sin 2\theta)$, we have $b_2' = 0$.

$\tau_{r\theta} = \frac{a_0' + 2\ d_2' \cos 2\theta}{r^2}$. Using the boundary condition $(\tau_{r\theta})_{\theta = \pm\alpha} = 0$, we have $a_0' = -2\ d_2' \cos 2\alpha$. To determine d_2', consider a sector of the wedge of radius r as free body. The resisting moment is due to the shear, since the radial forces go through the vertex, and their moment is zero. Summing moments about the vertex,

(7) $M = 2 \int_0^\alpha 2 \frac{d_2'}{r^2}(\cos 2\theta - \cos 2\alpha)\ r\ d\theta\ r$, from which

$$d_2' = -\frac{M}{2(\sin 2\alpha - 2\alpha \cos 2\alpha)}.$$

Using the values of the constants thus obtained, the stresses at any point are:

(8) $$\sigma_\theta = 0; \quad \sigma_r = -\frac{2\ M \sin 2\theta}{k\ r^2};$$

$\tau_{r\theta} = \frac{M}{k\ r^2}(\cos 2\theta - \cos 2\alpha)$, where $k = \sin 2\alpha - 2\alpha \cos 2\alpha$.

By superposition of the concentrated load at the vertex perpendicular to the axis upon the concentrated load at the vertex in the direction of the axis, we obtain the solution for a load inclined at any angle to the axis. By combining this with the solution for the moment about the vertex, we obtain the stresses in the frustrum of a wedge, subjected to a concentrated load at the free end.[3] The loading is shown in Figure 3. Equal and opposite forces P are introduced at the vertex of the completed wedge, and the forces are then

3. See Akira Miura, "Spannungskurven in rechteckigen und keilformigen Tragern", 1928.

grouped to give a concentrated force P at the vertex and a moment of magnitude -Pa sin γ, where 'a' is the perpendicular distance from the vertex to the free end of the frustrum. It should be noted, however, that the stresses across the free end of the frustrum due to this concentrated load P at the vertex of the completed wedge are not exactly neutralized by the moment M = -Pa sin γ, acting about the vertex. Hence our results will hold only for considerable distances from the free end of the frustrum. However, calculations show that for values of 2α less than approximately 45°, the maximum values of the σ_r stress occur at a distance from the free end greater than the depth of the free end, which is 2a tan α, so that our results are valid at that section.

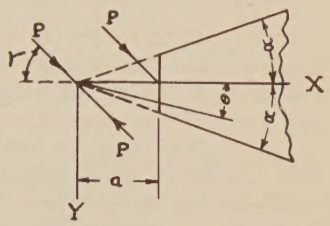

Figure 3

By increasing α to 90°, we have a semi-infinite plate, and the foregoing results are in agreement for the concentrated load at the boundary, and for the moment acting at the boundary.

To determine the stresses due to uniformly distributed normal loading, as shown in Figure 4, we choose the terms from Equation (4) which do not contain any power of r, and apply the boundary conditions. We find: $a_2 = 0$; $b_0 = \frac{p - q}{4}$;

$$c_2 = \frac{p + q}{4k}; \quad d_0 = -\frac{(p+q)\cos 2\alpha}{2k};$$

$$k = \sin 2\alpha - 2\alpha \cos 2\alpha.$$

This gives stresses as follows:

$$(9)\quad \sigma_r = \frac{p - q}{2} - \frac{p + q}{2k} (2\theta \cos 2\alpha + \sin 2\theta);$$

$$\sigma_\theta = \frac{p - q}{2} - \frac{p + q}{2k} (2\theta \cos 2\alpha - \sin 2\theta);$$

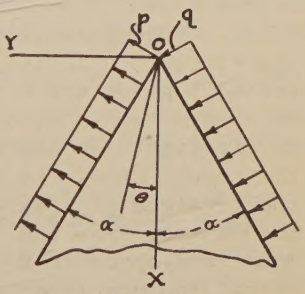

Figure 4

$$\tau_{r\theta} = -\frac{p+q}{2k}(\cos 2\theta - \cos 2\alpha).$$

When p is taken equal to zero, we have the infinite wedge with uniformly distributed normal load applied to one face.[4]

The results are of particular interest when applied to the semi-infinite plate. When we put $\alpha = \pi/2$, equation (9) becomes:

$$(10)\quad \sigma_r = \frac{p-q}{2} + \frac{p+q}{2\pi}(2\theta - \sin 2\theta)$$

$$\sigma_\theta = \frac{p-q}{2} + \frac{p+q}{2\pi}(2\theta + \sin 2\theta)$$

$$\tau_{r\theta} = -\frac{p+q}{2\pi}(1 + \cos 2\theta).$$

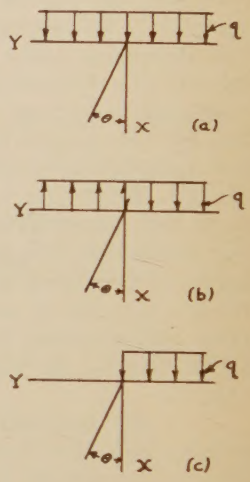

Figure 5

When p = -q, we have the case of a semi-infinite plate with uniformly distributed normal load of q per unit length applied at the boundary, as shown in Figure 5a. Hence:

$$(11)\quad \sigma_r = \sigma_\theta = -q;\ \tau_{r\theta} = 0.$$

When q = p, we have a semi-infinite plate with uniformly distributed normal load of q per unit length applied at the boundary, abruptly changing sign at the origin, as shown in Figure 5b. Equation (10) becomes:

$$(12)\quad \sigma_r = \frac{q}{\pi}(2\theta - \sin 2\theta)$$

$$\sigma_\theta = \frac{q}{\pi}(2\theta + \sin 2\theta)$$

$$\tau_{r\theta} = -\frac{q}{\pi}(1 + \cos 2\theta).$$

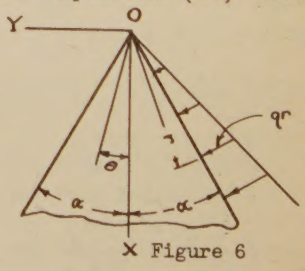

Figure 6

4. This case is given in Timoshenko, Theory of Elasticity, 1932, with original references.

When p = 0 in (10), we have the solution for the semi-infinite plate with a uniformly distributed load on one side of the origin, as shown in Figure 5c. In all these cases, the results agree with results obtained by other methods.

For triangular distribution of load, increasing uniformly from the vertex, as shown in Figure 6, we choose the terms of Equation (4) which contain the first power of r.[5] The stresses, expressed in a symmetrical form, are:

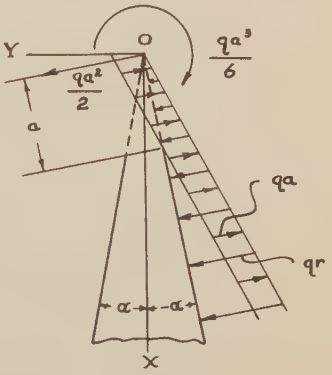

Figure 7

$$(13)\quad \sigma_r = \frac{q\,r}{4}\left[\frac{\sin^2\theta\,\cos\theta}{\cos^3\alpha} - \frac{\cos\theta}{\cos\alpha} + \frac{\sin\theta}{\sin\alpha} - \frac{\sin\theta\,\cos^2\theta}{\sin^3\alpha}\right];$$

$$\sigma_\theta = \frac{q\,r}{4}\left[\frac{\cos^3\theta}{\cos^3\alpha} - \frac{3\cos\theta}{\cos\alpha} + \frac{3\sin\theta}{\sin\alpha} - \frac{\sin^3\theta}{\sin^3\alpha}\right];$$

$$\tau_{r\theta} = \frac{q\,r}{4}\left[\frac{\sin\theta\,\cos^2\theta}{\cos^3\alpha} - \frac{\sin\theta}{\cos\alpha} - \frac{\cos\theta}{\sin\alpha} + \frac{\cos\theta\,\sin^2\theta}{\sin^3\alpha}\right].$$

Valid results are not obtained by attempting to extend these results to the semi-infinite plate. Since the stresses all contain the factor $\frac{1}{\cos^3\alpha}$, we get infinite stresses when $\alpha = \pi/2$. Changing the stress components into rectangular form does not remove this disturbing factor.

When we use the terms of Equation 4 which contain r^2, (or higher powers of r), the solution for the wedge is easily obtained, but we find similar difficulty in attempting to extend the solution to apply to the

5. See P. Fillunger, Zeitschr. f. Math. und Phys., Vol. 60, 1912.

semi-infinite plate.

Triangular distribution of normal load on one face of a truncated wedge may be obtained by superposition of loads shown in Figure 7, as follows:

(a) Triangular distribution of load qr, beginning at the vertex, and acting in the positive direction of θ.

(b) Uniformly distributed normal load of intensity qa, acting in the negative direction of θ.

(c) A concentrated load of magnitude $\frac{q\,a^2}{2}$, perpendicular to the edge, applied at the vertex, and acting in the positive direction of θ.

(d) A moment of magnitude $\frac{q\,a^3}{6}$, acting about the vertex in the positive direction of θ.

The general case of uniformly distributed shearing forces acting on the two faces of the wedge is shown in Figure 8. The directions of s_1 and s_2 as shown are positive. We choose the terms of equation (4) which are of zero degree in r, and apply the boundary conditions, $(\sigma_\theta)_{\theta=-\alpha} = (\sigma_\theta)_{\theta=\alpha} = 0$; $(\tau_{r\theta})_{\theta=-\alpha} = s_1$; $(\tau_{r\theta})_{\theta=\alpha} = s_2$. The stresses are:

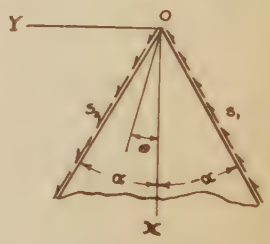

Figure 8

$$(14)\quad \sigma_r = -\frac{s_2 - s_1}{2 \sin 2\alpha}(\cos 2\theta + \cos 2\alpha) - \frac{s_2 + s_1}{k}(\alpha \sin 2\theta + \theta \sin 2\alpha);$$

$$\sigma_\theta = \frac{s_2 - s_1}{2 \sin 2\alpha}(\cos 2\theta - \cos 2\alpha) + \frac{s_2 + s_1}{k}(\alpha \sin 2\theta - \theta \sin 2\alpha);$$

$$\tau_{r\theta} = \frac{s_2 - s_1}{2 \sin 2\alpha} \sin 2\theta + \frac{s_2 + s_1}{2k}(\sin 2\alpha - 2\alpha \cos 2\theta);$$

where $k = \sin 2\alpha - 2\alpha \cos 2\alpha$.

By setting $s_2 = 0$, we have a uniformly distributed shearing load applied to one face of the wedge.

When $s_1 = s_2$, we have a uniformly distributed shearing load of equal intensity on both faces of the wedge, acting toward the vertex on one face, and away from the vertex on the other face. We find:

$$(15) \qquad \sigma_r = -\frac{2 s_1}{k}(\alpha \sin 2\theta + \theta \sin 2\alpha);$$

$$\sigma_\theta = \frac{2 s_1}{k}(\alpha \sin 2\theta - \theta \sin 2\alpha);$$

$$\tau_{r\theta} = \frac{s_1}{k}(\sin 2\alpha - 2\alpha \cos 2\theta).$$

Applied to the semi-infinite plate, this corresponds to uniformly distributed forces acting in one direction along the boundary, and equation (15) becomes:

$$(16) \quad \sigma_r = -s_1 \sin 2\theta; \ \sigma_\theta = s_1 \sin 2\theta; \ \tau_{r\theta} = -s_1 \cos 2\theta.$$

These are further simplified by changing into rectangular components, which gives:

$$(17) \quad \sigma_x = \sigma_y = 0; \ \tau_{xy} = -s_1,$$

and agrees with results obtained by other methods.[6]

When $s_2 = -s_1$, the shearing loads both act toward the vertex, and equation (14) becomes:

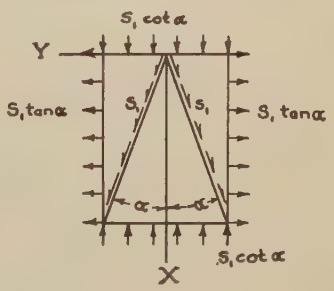

Figure 9

$$(18) \quad \sigma_r = \frac{s_1}{\sin 2\alpha}(\cos 2\theta + \cos 2\alpha);$$

$$\sigma_\theta = -\frac{s_1}{\sin 2\alpha}(\cos 2\theta - \cos 2\alpha);$$

$$\tau_{r\theta} = -\frac{s_1}{\sin 2\alpha} \sin 2\theta.$$

6. See Love, Theory of Elasticity, page 214.

It is interesting to note that the rectangular stress components obtained from Equation (18) are:

(19) $\sigma_x = s_1 \cot \alpha$; $\sigma_y = s_1 \tan \alpha$; $\tau_{xy} = 0$.

We thus see that the rectangular components of stress are constant throughout the wedge. This is a condition quite different from that which we have in the case of a load concentrated at the vertex, acting in the direction of the wedge axis. In that case, the stresses were proportional to powers of $\sin \theta$ and $\cos \theta$, and inversely proportional to x, while here the rectangular components of stress are independent of both x and θ. Physically, Equation (19) means that a wedge can be cut from a rectangular plate, which is loaded at the edges as shown in Figure 9, in such a manner that only shearing stresses are acting on the sides of the wedge.

Equations (16) and (17) give the stresses for the only case where satisfactory results are obtained by going from the wedge to the semi-infinite plate, for shearing loads. In the other cases, it is seen that the $\sin 2\alpha$ factor in the denominator gives infinite stresses for σ_y. Physically, Equation (17) means that rectangles can be cut from a semi-infinite plate loaded in this fashion so that only shearing forces are acting on the sides of the rectangles. The sides of the rectangles will be parallel and perpendicular to the edge of the semi-infinite plate. Superposition of the solutions for uniformly distributed normal and shearing loads gives the solution for uniformly distributed inclined load on the boundary of the wedge.

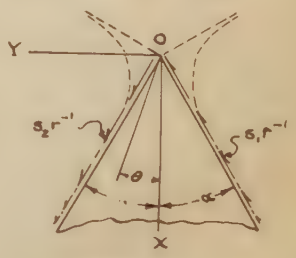

Figure 10

So far, we have taken account only of the terms of the general stress-function from which we may derive σ_θ or $\tau_{r\theta}$ stresses involving powers of r of zero degree or greater; of the concentrated load at the vertex, involving r^{-1} in the σ_r stress; and of the moment acting about the vertex of the wedge, involving r^{-2} in the σ_r and $\tau_{r\theta}$ stresses. Consider the other terms of the stress function which give stresses involving r^{-1} power. They are:

(20) $\varphi = r \log r \, (b_1' \cos \theta + d_1' \sin \theta)$.

The stresses calculated from this stress-function are:

(21) $\sigma_r = \sigma_\theta = r^{-1}(b_1' \cos \theta + d_1' \sin \theta)$;

$\tau_{r\theta} = r^{-1} (b_1' \sin \theta - d_1' \cos \theta)$.

If we use as boundary conditions:

$$(\tau_{r\theta})_{\theta=-\alpha} = s_1 r^{-1}; \; (\tau_{r\theta})_{\theta=\alpha} = s_2 r^{-1},$$

we find:

(22) $b_1' = \frac{s_2 - s_1}{2 \sin \alpha}; \; d_1' = -\frac{s_2 + s_1}{2 \cos \alpha}$.

When we substitute these constants into the expression for σ_θ, we see that the boundary is not free of normal stresses, but that certain prescribed normal forces corresponding to the applied shearing forces must be present. The stresses are:

(23) $\sigma_\theta = \sigma_r = r^{-1}\left(\frac{s_2 - s_1}{2 \sin \alpha} \cos \theta - \frac{s_2 + s_1}{2 \cos \alpha} \sin \theta\right)$;

$$\tau_{r\theta} = r^{-1}\left(\frac{s_2 - s_1}{2 \sin \alpha} \sin \theta + \frac{s_2 + s_1}{2 \cos \alpha} \cos \theta\right).$$

If we had assigned values to the normal forces at the boundary, we would have a similar condition in regard to the shearing forces at the boundary. As in the case of a concentrated load at the vertex, and of moment acting about the vertex, the stresses become infinite as r approaches zero, so we must consider a small sector at the vertex as being in a state of plasticity, and not conforming to our elastic theory.

In Equation (22), having either b_1' or d_1' equal to zero simply means that we have a special relation between s_2 and s_1. For $d_1' = 0$, $s_2 = -s_1$, and the shearing forces are of equal intensity at equal distances from the vertex, acting along the faces toward the vertex. The corresponding normal forces are:

$$(24)\quad (\sigma_\theta)_{\theta=\pm\alpha} = r^{-1} s_1 \frac{1 - \cos 2\alpha}{\sin 2\alpha} = - r^{-1} s_1 \cot \alpha.$$

If these normal forces are combined with the corresponding shearing forces, we have resultant forces of intensity $r^{-1} s_1 \csc \alpha$, perpendicular to the wedge axis, acting on the faces of the wedge, toward the axis.

The terms involving r^{-2} give stresses:

$$(25)\quad \sigma_\theta = - a_0 r^{-2};$$

$$\sigma_r = r^{-2}(a_0 - 4b_2' \cos 2\theta - 4 d_2' \sin 2\theta);$$

$$\tau_{r\theta} = r^{-2}(a_0' - 2 b_2' \sin 2\theta + 2 d_2' \cos 2\theta).$$

By putting $a_0 = a_0' = 0$, we have a case similar to the preceding. The term $a_0 r^{-2}$ is obtained from the term $a_0 \log r$ in the general stress function. It does not contain θ, hence represents stress distribution symmetrical with respect to an axis through the origin, perpendicular to the plane of the coordinate axes. The boundary of the wedge is such that it is not possible to use this term.

Of more interest is the term $a_0' r^{-2}$, which is obtained from the term $a_0' \theta$ in the general stress function. This gives:

$$(26)\quad \sigma_\theta = \sigma_r = 0;\ \tau_{r\theta} = \frac{a_0'}{r^2}.$$

We thus have a condition of pure shear, independent of θ, and dependent only on r^{-2}. This condition results

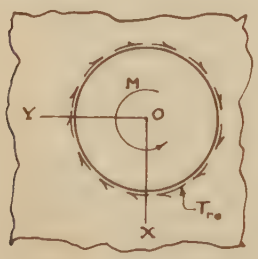

Figure 11

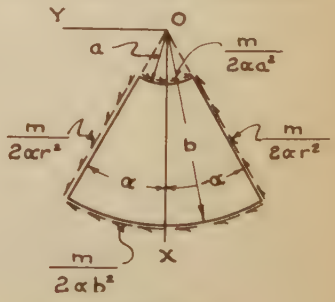

Figure 12

from a moment acting about the origin in the plane of an infinite plate, as shown in Figure 11. The constant a_0' can be determined by integration, summing up the moment, about the origin, of stresses acting on the elements r d θ, and equating to the negative applied moment. We thus have:

$$\int_0^{2\pi} \frac{a_0'}{r^2} r^2 \, d\theta = -M, \quad a_0' = -\frac{M}{2\pi}, \text{and}$$

$$\tau_{r\theta} = -\frac{M}{2\pi r^2}. \tag{27}$$

We may apply this to the wedge, by taking the limits of integration between $-\alpha$ and α. When this is done, the stresses are:

$$\sigma_r = \sigma_\theta = 0; \tau_{r\theta} = -\frac{M}{2\alpha r^2}. \tag{28}$$

This should not be confused with the case of pure moment about the vertex of the wedge, the results of which are given in equation (8). Associated with the moment as given here will be shearing forces on the faces of the wedge of magnitude $-\frac{M}{2\alpha r^2}$. If we consider a small sector near the vertex removed, to which the equations based on elastic theory do not apply, we have a blunted wedge, with boundary loads as shown in Figure 12.

Other negative integral powers of r may be treated in a manner similar to the cases considered above.